DESCRIPTIVE THEORY OF THE CHAOS : Attractors of trajectories and their basins

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How complicated is the one-dimensional chaos: descriptive theory of chaos

We consider one-dimensional dynamical systems given by maps $f \in C^0(I, I)$ with I being a closed interval under the condition that the topological entropy $h(f)$ is positive.

Most recently, the book [1] of Sylvie Ruette appeared, the aim of which is "to survey the relations between the various kinds of chaos and related notions for continuous interval maps". However, the book does not even mention the research of the talk author published back in the sixties of the past century [2-4]. The research results showed a huge variety of the trajectory attractors (i.e., ω-limit sets) and the more complexity of their attraction basins, pointing to a very intricate interweaving of different basins. All this gives a good idea about the complexity of the one-dimensional chaos.

In [2], using the descriptive sets theory, it was proved that even basins of the simplest attractors – cycles – can be very complex, namely, can be a set of the third class in the Baire classification. Later in [3], it was shown that such situation is typical, namely, even for quadratic maps, the basin of any attractor that contains a cycle and is not maximal or locally maximal (which is typical where $h(f) > 0$), is a set of the third Baire class.

In [4], the properties of the set of all trajectory attractors of f partially ordered by the set-theoretic inclusion, were formulated. If $h(f) > 0$, then in this set there exists at least one maximal attractor $\mathcal{A}$, that contains cycles and so contains continuum many of locally maximal attractors other than cycles (and being Cantor sets); each from these locally maximal attractors contains continuum many of minimal attractors other than cycles (and hence being Cantor sets). It remains to add that the basin of every attractor contained in $\mathcal{A}$ is dense on $\mathcal{A}$. The proofs of all statements can be found in the author's thesis from 1966 and in [5, sect. 4.1].

1. S.Ruette, *Chaos on the interval*, Amer. Math. Soc., ser. University lecture **67**, 2017.
2. A.N.Sharkovsky, *A classification of fixed points*, Amer. Math. Soc. Transl. (2) **97**, 1970, 159-179 (transl. from Ukrain. Mat. Zh. **17**(5), 1965, 80-95).
3. A.N.Sharkovsky, *Behavior of a mapping in the neighborhood of an atrracting set*, Amer. Math. Soc. Transl. (2) **97**, 1970, 227-258 (transl. from Ukrain. Mat. Zh. **18**(2), 1966, 60-83).
4. A.N.Sharkovsky, *Partially ordered system of attracting sets*, Soviet Math. Dokl. **7**, 1966, 1384-1386 (transl. from Dokl. Akad. Nauk SSSR **170**(5), 1966, 1276-1278).
5. A.N.Sharkovsky, *Attractors of trajectories and their basins*, Naukova Dumka, Kiev, 2013, 320p. (in Russian).

Short summary of the talk :

Descriptive theory of sets is a classical section of mathematics, which arose at the beginning of the last century.

The talk proposes

the basis of the descriptive theory of deterministic chaos :

Dynamical system if its topological entropy is positive

1) has a lot of different attractors of trajectories, namely, the continuum of attractors;

2) basins of most attractors have a very complex structure, namely, they are sets of the 3rd class in the terminology of the descriptive theory of sets;

3) basins of different attractors are very intertwined and they can not be separated from each other by open or closed sets, but only by sets of the 2nd class of complexity, and

in the space of all closed subsets of the state space (with the Hausdorff metric), the set of all attractors is an attractor net (network, grid) whose cells are formed by Cantor sets (whose points are themselves attractors of the dynamical system).

We consider dynamical systems on a compact X, generated by a continuous map $f : X \to X$, mainly in the case of when X is an interval $I \subset R$.

The asymptotic behavior of every trajectory $f^i(x)$, $i = 0, 1, 2, ...,$ $x \in X$, is usually determined through the so-called ω-limit set, or, more simply, *the attractor* of this trajectory – the invariant closed set $\mathcal{A}_x = \bigcap_{m>0} \overline{\{\bigcup_{i>m} f^i(x)\}}$, which attracts the trajectory when the time goes to infinity: for any of its vicinity U, there exists $i_0 = i_0(U)$ such that $f^i(x) \in U$ when $i \geq i_0$.

Each of the sets $\mathcal{A}_x$, $x \in X$, can be as an attractor for many trajectories. The set of all trajectories attracted by the same attractor is called *the basin of this attractor*: if $\mathcal{A}$ is an attractor, then $\mathfrak{B}(\mathcal{A}) = \{x \in X \mid \mathcal{A}_x = \mathcal{A}\}$ is the basin of the attractor $\mathcal{A}$.

Most of the results presented in this talk were obtained and published in the 60th of the last century, but even now, it seems, they are little known, although all of them were translated into English at the same time.

[1] *On attracting and attracted sets*, Soviet Math. Dokl. **6**, 1965, 268-270 (transl. from Dokl. Akad. Nauk SSSR **160**, 1965, 1036-1038).

[2] *A classification of fixed points*, Amer. Math. Soc. Transl. (2) **97**, 1970, 159-179 (transl. from Ukrain. Mat. Zh. **17**(5), 1965, 80-95).

[3] *Behavior of a mapping in the neighborhood of an atrracting set*, Amer. Math. Soc. Transl. (2) **97**, 1970, 227-258 (transl. from Ukrain. Mat. Zh. **18**(2), 1966, 60-83).

[4] *Partially ordered system of attracting sets*, Soviet Math. Dokl. **7**, 1966, 1384-1386 (transl. from Dokl. Akad. Nauk SSSR **170**, 1966, 1276-1278).

[5] *Structure of an endomorphisms on ω-limit sets*, Intern. Math. Congress (Moscow,1966), Sect.6, Abstracts, 51.

[6] *Attractors of trajectories and their basins*, Naukova Dumka, Kiev, 2013, 320p. (in Russian).

H e r e :

attracting set = attractor of a trajectory = ω-limit set of a trajectory

attracted set = basin of an attractor = subset of the phase space consisting of all trajectories with the same ω-limit set

[7] Bondarchuk V.S., Sharkovsky A.N., *Reconstructibility of expanding endomorphisms from the system of omega-limit sets* [in Russian], in Dynamical systems and questions on the stability of the solutions of differential equations, Inst. Mat. Akad. Nauk USSR, Kiev, 1973, 28-34.

[8] Bondarchuk V.S., Sharkovsky A.N., *The partially ordered system of omega-limit sets of expanding endomorphisms* [in Russian], ibid., 128-164.

[9] Bondarchuk V.S., *Invariant sets of smooth dynamical systems*, PhD Thesis, 1974.

[10] Shakovsky A.N. *How complicated can be one-dimensional dynamical systems: descriptive estimates of sets*, Dynamical systems and ergodic theory (Warsaw, 1986), Banach Center Publ., **23**, Warsaw, 1989, 447-453.

[11] Sivak A.G., *Descriptive estimates for statistically limit sets of dynamical systems,* in *Dynamical Systems and Turbulence* [in Russian], Inst. Math., Kiev, 1989, 100-102.

[12] Sivak A.G., *On the structure of the set of trajectories generating an invariant measure,* in *Dynamical Systems and Nonlinear Phenomena* [in Russian], Inst. Math., Kiev, 1990, 39-43.

[13] Sivak A.G., *σ-Attractors of trajectories and their basins,* Addition in [5], sect.7, 281-310.

[14] Sharkovsky A.N., Sivak A.G., *Basins of attractors of trajectories,* J. Difference Equations and Appl., **22**(2), 2016, 159-163.

Now it's well known that in one-dimensional dynamical systems the chaos exists when the topological entropy of a system is positive. Also it is well known that for one-dimensional systems the following is true:

For the dynamical system given by a map $f \in C^0(I, I)$, where I is a closed interval, the following statements are equivalent:

(1) the topological entropy is positive, $h(f) > 0$;
(2) f has a cycle of period $\neq 2^i$, $i \geq 0$;

(3) f has a homoclinic trajectory;

(4) there are $m \geq 1$ and closed intervals $J, K \subset I$ such that $f^m J \cap f^m K \subset J \cup K$;

and each of them implies two more equivalent statements:

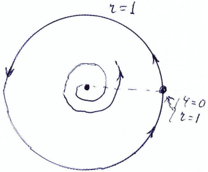
(5) there are $x, y \in I$ such that $\lim_{i\to\infty} \sup \rho[f^i(x), f^i(y)] > 0$ and $\lim_{i\to\infty} \inf \rho[f^i(x), f^i(y)] = 0$;

(6) there is a continuum of trajectory pairs with the property (5).

The word *chaos* as a mathematical term first appeared in the article "Period three implies chaos" by Li, Yorke (Monthly **103**, 1975), and exactly the property (6) was used there as the main in determining chaos.

However, the property (6), in general, can not be decisive for the characterization of chaos. For example, the simple system of equations $\dot{r} = r(1-r)$, $\dot{\varphi} = \alpha$ (in the polar coordinates) has the periodic trajectory $r = 1$ which is an attractor for this system. After multiplying the both right-hand sides of these equations by the factor $h(r,\varphi)$ such that $h(1,0) = 0$, $h(r,\varphi) > 0$ for others r,φ, almost each pair of trajectories will have the property (5), i.e., the new system will have the property (6). Indeed, after multiplying by h the phase portrait of the system remains unchanged, only the periodic trajectory $r = 1$ is transformed into a homoclinic trajectory to the singular point $(1,0)$. Namely the presence of the singular point, where the velocity of motion is equal to 0, leads to this effect: when a trajectory approaches to this point, the velocity decreases significantly, and when the trajectory moves away from it, the velocity increases again. Of course, there is no chaos in this system.

$$\dot{r} = r(1-r) \qquad \times\, \forall h(r,\varphi)$$
$$\dot{\varphi} = \alpha \qquad \begin{cases} h(1,0) = 0 \\ h(r,\varphi) > 0 \end{cases}$$
$$h(r,\varphi) = (r-1)^2 + \varphi^2$$



In the theory of dynamical systems, along with open sets (for example, basins of sinks, wandering sets) and closed sets (ω-limit sets, nonwandering sets, centers of dynamical systems), sets with more complicated structure are considered.

There appear F_σ sets, which are unions of no more than countably many closed sets (e.g. the set of all periodic points), G_δ sets, which are intersections of no more than countably many open sets (e.g. the set of all transitive points of transitive systems), $F_{\sigma\delta}$ sets, which are intersections of no more than countably many F_σ sets, etc.

We also use Baire's classification of sets according to which open sets and closed sets together with all sets being both F_σ and G_δ constitute the first class. The second class consists of sets that are either F_σ or G_δ but not both, and sets that are simultaneously $F_{\sigma\delta}$ and $G_{\delta\sigma}$ but do not belong to the first class. The third class consists of sets being either $F_{\sigma\delta}$ or $G_{\delta\sigma}$ but not both, and sets that ... Further classes are defined in a similar way.

Usually descriptive upper estimates are obtained relatively easy, even for dynamical systems on spaces X with countable basis of its topology, and in [1], such upper estimates for systems on arbitrary compacts have been obtained. Namely:

(a) if an attractor $\mathcal{A}$ is maximal, i.e. there is no attractors $\tilde{A} \supset \mathcal{A}$, then the basin $\mathcal{B}(\mathcal{A})$ is a G_δ set;

(b) if an attractor $\mathcal{A}$ is locally maximal, i.e. there exists a neighborhood of $\mathcal{A}$, not containing attractors $\tilde{A} \supset \mathcal{A}$, then the basin $\mathcal{B}(\mathcal{A})$ is both a $F_{\sigma\delta}$ set and a $G_{\delta\sigma}$ set;

(c) in any case, basin $\mathcal{B}(\mathcal{A})$ is (no more complex than) a $F_{\sigma\delta}$ set in X, i.e. it always can be represented as an intersection of no more than countably many unions of no more than countably many closed sets.

But the proof of the accessibility of these estimates at least for a certain class of systems, and thus, the proof of the complex interlacing of the basins of investigated attractors is really a very complicated problem even in dimension one ...

Nevertheless, as it turned out, all these estimates are accessible for one-dimensional systems when f has a cycle of period $\neq 2^i$. Namely, it was shown in [2-4] that in this case there exists a maximal attractor A_{max} containing a cycle as well as continuum many attractors of kind (c); the basin of each such (of kind (c)) attractor is a third class set, i.e., is a $F_{\sigma\delta}$ set but not a $G_{\delta\sigma}$ set. This means that

1) here we have the very complex curved interlaced trajectories with different asymptotic behavior, and

2) from the viewpoint of descriptive theory of sets, one-dimensional chaos is as complex as is many-dimensional or even infinity-dimensional chaos.

The arithmetic Baire's example of a third class set.

Let J be the irrational points set of the interval $(0,1)$. For every point of J, there corresponds a unique continued fraction of the form

$$\cfrac{1}{n_1+\cfrac{1}{n_2+\cfrac{1}{n_3+\ldots}}}.$$

The Baire set $\mathbb{B}$ of third class consists of points from J for which $n_j \to \infty$.

The same example from the "point of view" of dynamical systems: Let the map $g: J \to J$ be given by $g: x \mapsto \{1/x\}$, where $\{\cdot\}$ is for the fractional part of a number. The map g is continuous on J (in the metrics for continued fractions) and $g(J) = J$. Indeed, if $x = \frac{1}{n_1+\frac{1}{n_2+\frac{1}{n_3+\ldots}}}$, then $g(x) = \frac{1}{n_2+\frac{1}{n_3+\ldots}}$.

Thus, the Baire set $\mathbb{B}$ is constituted by the points $x \in J$ for which $g^j x \to 0$ as $j \to \infty$, i.e., in our notations, $\mathbb{B}$ is just the set $\mathfrak{B}(\{0\})$ – the basin of the point $x = 0$.

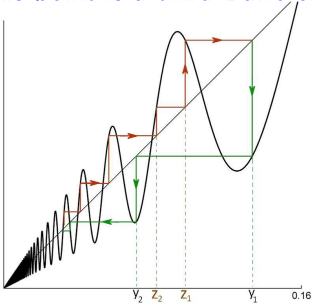
The Baire criterion for belonging a set to the third class.

Let $p_{j_1 j_2 \dots j_k}$ be perfect nowhere dense on R or J sets, and

1) $p_{j_1 \dots j_{k-1} j_k} \subset p_{j_1 \dots j_{k-1}}$,
2) $p_{j_1 \dots j_{k-1} j_k}$ is nowhere dense on $p_{j_1, \dots, j_{k-1}}$,
3) $\bigcup\limits_{j_k=1}^{\infty} p_{j_1 \dots j_{k-1} j_k}$ is everywhere dense on $p_{j_1 \dots j_{k-1}}$.

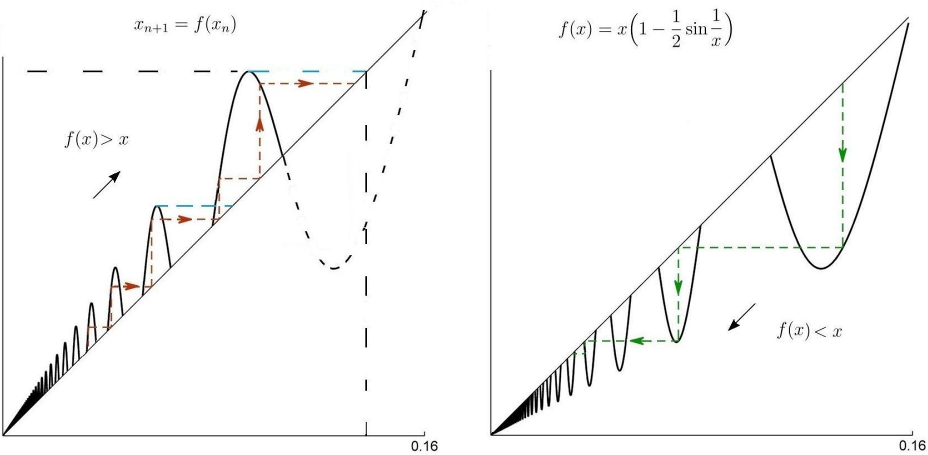
The set $P = \bigcap\limits_{k=1}^{\infty} \bigcup\limits_{j_1, \dots, j_k=1}^{\infty} p_{j_1 \dots j_k}$ is of Baire's third class.

Theorem. *If a map has an attracting-repulsing fixed point, then the basin of this point is a third Baire class set.*



Road to chaos through the "creeping" feedback (nonsmooth realization)

We can see in this figure how the repulsion from the fixed point ($x = 0$) and the attraction to it occur ("creeping" feedback). It remains to show that the set of points x for which $f^i(x) \to 0$ when $i \to \infty$ can be represented as an union of two sets, namely, a set that satisfies the Baire criterion for being in the third class and a set of a Baire class ≤ 2. This complicated problem has been solved in [2].


$x_{n+1} = f(x_n)$
$f(x) > x$
0.16
$f(x) = x\left(1 - \frac{1}{2}\sin\frac{1}{x}\right)$
$f(x) < x$
0.16

Theorem. *If an attractor $\mathcal{A}$ is not maximal or locally maximal and contains a cycle, then the basin $\mathcal{B}(\mathcal{A})$ is a third Baire's class set.*

The theorem proof is given in [3] and is based on the proof of the corresponding theorem for the cycles in [2].

For one-dimensional systems, **only the irreversibility of f gives** an opportunity for the **feedback, which opens the way to chaos.**

In our case, there exists a maximal attractor $\mathcal{A}_{max} \supset \mathcal{A}$ which contains points x such that $f^{-1}(x)$ consists of at least two points and thus there arises a "fast feedback" on $\mathcal{A}_{max}$. Here we obtain the "fast feedback" both for cycles and for locally maximal attractors. Nevertheless, the "creeping feedback" remains decisive for attractors that are not locally maximal.

The simplest example of such an attractor can be a homoclinic trajectory as well as a cycle to which it directs ...

How does this "creeping feedback" occur?

On the attractor $\mathcal{A}_{max}$, the basin of every attractor $\mathcal{A} \subseteq \mathcal{A}_{max}$ is a dense set, $\overline{\mathcal{B}(\mathcal{A}) \cap \mathcal{A}_{max}} = \mathcal{A}_{max}$.

The basin $\mathcal{B}(\mathcal{A}_{max})$ is a G_δ set of the second Baire class, the basin of every locally maximal attractor is both a $F_{\sigma\delta}$ and a $G_{\delta\sigma}$ set and hence it is of the second Baire class.

For any attractor $\mathcal{A} \subset \mathcal{A}_{max}$ that is not locally maximal and contains a cycle, the basin $\mathcal{B}(\mathcal{A})$ is a $F_{\sigma\delta}$ set of the third Baire class. At the same time, basins of any two such attractors $\mathcal{A}', \mathcal{A}''$, $\mathcal{A}' \cap \mathcal{A}'' = \emptyset$, are separated by sets of the second Baire class: there exist two locally maximal atractors $\mathcal{A}'_{lmax} \supset \mathcal{A}'$ and $\mathcal{A}''_{lmax} \supset \mathcal{A}''$ such that $\mathcal{A}'_{lmax} \cap \mathcal{A}''_{lmax} = \emptyset$, and then the F_σ sets

$$\mathcal{B}' = \bigcup_{i=0}^{\infty} f^{-i}(\mathcal{A}'_{lmax}) = \bigcup_{\mathcal{A} \subseteq \mathcal{A}'_{lmax}} \mathcal{B}(\mathcal{A}) \quad \text{and}$$

$$\mathcal{B}'' = \bigcup_{i=0}^{\infty} f^{-i}(\mathcal{A}''_{lmax}) = \bigcup_{\mathcal{A} \subseteq \mathcal{A}''_{lmax}} \mathcal{B}(\mathcal{A})$$

separate the basins of $\mathcal{A}'$ and $\mathcal{A}''$.

Of course, the information about the attractors themselves and their interrelations must form an essential part of the descriptive theory of chaos.

In [4] and [6, sec.4], the families $\mathfrak{M}$ and $\mathfrak{M}'$ of all attractors and of all locally maximal attractors contained in $\mathcal{A}_{max}$, respectively, are considered.

$\mathfrak{M}$ contains a continuum of locally maximal attractors, other than cycles; each of them is a Cantor set, on which periodic points are everywhere dense.

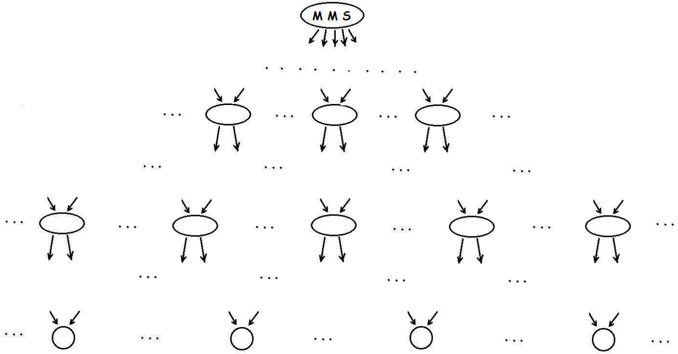
$\mathfrak{M}$ contains a continuum of minimal attractors, different from cycles, and, hence, all of them are Cantor sets.

There is a natural partial order in $\mathfrak{M}$, namely:
if $\mathcal{A}' \subset \mathcal{A}$, then $\mathcal{A}'$ precedes $\mathcal{A}$ in $\mathfrak{M}$.

The maximal attractor $\mathcal{A}_{max}$ and every locally maximal attractor have no direct predecessor in any maximum chain. Each attractor of such a chain, other than $\mathcal{A}_{max}$, has a continuum of direct successions.

Every maximal chain from $\mathfrak{M}'$ contains a countable number of locally maximal attractors and is similar to the set of rational points:

$$\text{for each } \mathcal{A}' \subset \mathcal{A}'', \;\; \text{there exists } \mathcal{A}''' \;\; \text{such that } \mathcal{A}' \subset \mathcal{A}''' \subset \mathcal{A}'' \; ...$$



stands for a locally maximal omega-limit set which is a Cantor set

stands for a locally maximal omega-limit set which is a finite set (a cycle)

each arrow $\downarrow$ *replaces the sign* >

2^X

The family of all attractors $\mathfrak{M}$ when considered as a set in the space 2^X with the Hausdorff metric seems very interesting.

The family $\mathfrak{M}$ in the space 2^I forms a closed set [1996; Blokh, Bruckner, Humke, Smital]. This set is not dense on 2^I.

In the space 2^X, the family $\mathfrak{M}'$ and the family of all cycles $\mathfrak{P} \subset \mathfrak{M}$ form the sets which ars everywhere dense on the set $\mathfrak{M}$, i.e.

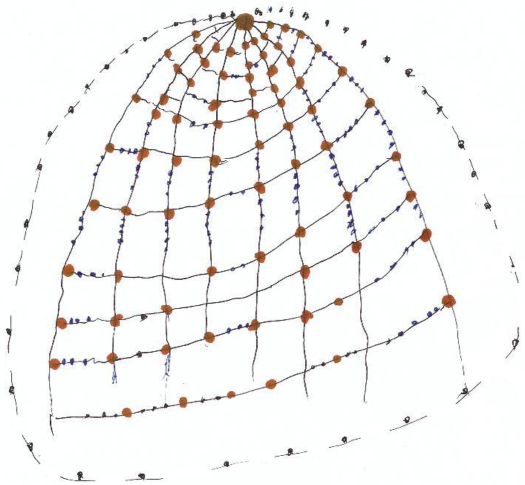
$$\overline{\mathfrak{P}} = \overline{\mathfrak{M}'} = \mathfrak{M}$$

Attractor net (network) of Cantor sets

Every maximal chain $\mathfrak{L}$ from $\mathfrak{M}' \setminus \mathfrak{P}$ after its closure in the Hausdorff metric, transforms into a Cantor set in 2^X, which begins at the point corresponding to the attractor $\mathcal{A}_{max}$ and finishes with a point corresponding to a minimal or almost minimal attractors, on which all or almost all trajectories are everywhere dense.

Various maximal chains from $\mathfrak{M}' \setminus \mathfrak{P}$ intersect at some locally maximal attractors, which leads in the space 2^X to the intersection of different Cantor sets at certain points and the formation (by Cantor sets) of a network whose nodes are just locally maximal attractors.

Hence, $\mathfrak{M}$ as a set in the space 2^X consists of the network of intertwined Cantor sets, which begins at the point corresponding to the attractor $\mathcal{A}_{max}$, and of a countable number of isolated points corresponding to cycles from $\mathfrak{P}$.



The set $\mathfrak{M}$ has a certain self-similarity. So, if $X = S^1$ and $f : x \mapsto 2x \mod 1$, then the following statement seems plausible: if $\mathcal{A}^* \in \mathfrak{M}'$ is a locally maximal attractor and $\mathfrak{M}_{\mathcal{A}^*} = \{\mathcal{A} \in \mathfrak{M} \mid \mathcal{A} \subseteq \mathcal{A}^*\}$, then on 2^X there exists a homeomorphism $\phi_{\mathcal{A}^*}$ such that $\phi_{\mathcal{A}^*}(\mathfrak{M}) = \mathfrak{M}_{\mathcal{A}^*}$.